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E- Baysian estimation of survival function of Topp Leone distribution based on type II censoring data

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Abstract*:* The content of this research is an estimate of the survival function of the Topp Leone distribution and its shape parameter based on type II control data. This was done by using the Standard Baysian method, the Expected Baysian method, and the Hierarchical Bysian method using quadratic loss functions and they were compared with the maximum possibility method and the comparison between the estimation methods was made using *the mean absolute relative error (MAPE).* comparison standard, where the simulation method was made by the Monte Carlo method to compare the results of the estimation methods and after taking different sample sizes small, medium and large. The results showed that the Bayesian . method is the best in Estimated to have the smallest square error.

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Keywords: Topp leone distribution. Bayesian method ,E-Bayesian method maximum possibility, second type control data.

1. Introduction:

Recent research and studies focused on estimating the survival function of failure distributions. These studies differed according to the methods used in estimating their importance in vital fields, especially in the field of medicine, as they are of great importance in studying the survival of organisms alive during a specified period of time. And one of the most important reasons for studying the survival function is that it determines the determinants of risk in the medical fields, as these studies led to the emergence of many methods, methods and data in the analysis that differ according to the nature of the studies under research and In our research, Censoring data type II was adopted to estimate the survival function of the Topp Leone distribution as well as the shape parameter of the distribution.

In 1955, researchers Topp and Leone proposed a simple continuous distribution in the form of the letter J. This distribution has attracted the attention of many statistical researchers because it is considered an alternative to the Beta distribution

The random variable t is a J-shaped distribution if the aggregate distribution function is as follows [9]:

$$F(x) = \begin{cases} 0 & t < 0 \\ \frac{t}{\theta}(2 - \frac{t}{\theta})^{\lambda} & 0 \le t \le \theta, \lambda > 0 \\ 1 & \theta < t \end{cases} \quad \dots\dots\dots (1)$$

Where the parameter θ is the measurement parameter and the parameter λ is the shape parameter.

As for the probability function of the distribution, it is given as:

$$f(x) = \frac{2\lambda}{\theta}\left(1 - \frac{t}{\theta}\right)\left(\frac{t}{\theta}(2 - \frac{t}{\theta})\right)^{\lambda} \qquad 0 \le t \le \theta, \lambda > 0 \qquad \dots\dots\dots (2)$$

The distribution is called J-shaped because $f(t) > 0$ and the first derivative of the distribution is negative $f(t)^{'} < 0$ and the second derivative of the distribution is positive $f(t)^{''} > 0$ for each $0 \le t \le \theta$ [3].

The researchers Topp and Leone gave the first, second, third and fourth moments, and the model was adapted as a failure model, and this distribution was not dealt with until 2003. The researchers Nadarajah and Kotz gave the failure rate function for the distribution as well as the additional general derivative formulas for central moments and moments. In addition, the formula and the estimator of the maximum possibility were derived For the parameter λ when the parameter θ is known and they have shown that the failure rate function is either constant or decreasing in a monotonous manner or increasing in a monotonous manner or being in the form of a U-shaped and they have proven that all cases are important in the practical side. When the failure function is in the form of U, this case applies to humans, where the death rate in infants is high due to congenital diseases and birth defects, then the death rate remains constant until the age of thirty and then rises again.

Failure models with a U-shaped failure rate function usually have three or more parameters, which in turn makes estimation and testing a problem unless large samples are available for research. But what distinguishes J-shaped distributions that have a U-shaped failure rate function on only two parameters is the shape parameter and the scaling parameter [10].

In our research, we will assume that the measurement parameter θ = 1, then the cumulative distribution function will become as follows:

$$F(x) = \begin{cases} 0 & t < 0 \\ t(2 - t)^{\lambda} & 0 \le t \le \theta, \lambda > 0 \\ 1 & t > 1 \end{cases} \quad \dots\dots\dots (3)$$

The probability function for the distribution is as follows:

$$f(x) = 2\lambda(1-t\ )(t(2-t))^{\lambda-1} \qquad 0 \le t \le 1, \lambda > 0 \qquad \dots\dots\dots(4)$$

So, the survival function for the distribution is as follows:

$$S(t) = 1 - F(t) \dots\dots\dots.(5)$$

$$S(t) = 1 - t(2-t)^{\lambda} \dots\dots\dots(6)$$

Our goal in this research will be to estimate the λ shape parameter of the distribution as well as the survival function $S(t)$

2-Methods of estimating

In this part of the research, the methods of estimating the shape parameter λ and the survival function $S(t)$ will be discussed, and these methods are firstly the maximum possibility method and compared with the standard *Bayesian* method and the Bayesian . method as well as the hierarchical *Bayesian* method

1-2 Maximum likelihood method (MLE)

One of the most important classical methods in the estimation process is the maximum possibility method because it has the property of stability (invariance property) as well as many important advantages.

In the data or monitoring system of the second type, the units are put under experiment, whatever they are, whether people or industrial experimental units with a number of n under observation with the life model test in this case according to the longevity of the experimental unit at time zero, and time can be expressed as a random variable that cannot be determined and is determined The number of units under observation is r so that $n > r$ the data consists of observations $(t_1, t_2, \dots, t_r)$ as these observations represent the ages of the units under test and this means that there is no information about the unit $(n-r)$ except for the units that Its lifetime is greater than t_r and the experiment ends when unit r fails.

The function of the function of the maximum possibility of observations in the presence of data of the second type for the distribution of Top Lyon was described by the scientist Cohen (1965) as follows [8]

$$L(\lambda/t) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(\lambda, t_i)[S(t_r)]^{n-r} \dots\dots\dots.(7)$$

$$L\left(\frac{\lambda}{t}\right)=\frac{n!}{(n-r)!}\prod_{i=1}^{r}(\beta(2-2t_i)(2t_i-t_i^2)^{\lambda-1}[1-(t_r(2-t_r))^{\lambda}]^{n-r}\ \dots\dots\dots(8)$$

$$(\lambda/t)=\frac{n!}{(n-r)!}\lambda^r 2^r\prod_{i=1}^{r}\frac{1-t_i}{t_i(2-t_i)}e^{-\lambda N}\ \dots\dots\dots(9)$$

$$L(\lambda,t)=vu(\lambda;\underline{t})e^{-\lambda N}\ \dots\dots\dots(10)$$

whereas

$$v=\frac{n!}{(n-r)!}2^r$$

$$\underline{t}=(t_1,t_2,\dots,t_r),\ \ u(\lambda;\underline{t})=\prod_{i=1}^{r}\frac{\lambda^r(1-t_i)}{t_i(2-t_i)}$$

$$N=-\sum_{i=1}^{r}log(2t_i+t_i^2)+(n-r)log((2t_r-t_r^2)$$

In order to obtain the maximum possible estimator for the parameter λ, equation 10 is derived after taking the natural logarithm of the equation, so the maximum possibility estimator is as follows [7]

$$\hat{\lambda}_{mle}=\frac{r}{N}\dots\dots\dots(11)$$

Then the estimator of the survival function of the Topp Leone distribution will be as follows, according to the stability property

$$\hat{S}_{mle}=1-t(2-t)^{\frac{r}{N}}\dots\dots\dots(12)$$

2-2. Standard Bayesian method

In this part of the research, the standard Bayes method of estimating will be reviewed, which states that the parameters to be estimated are random variables

that can be formulated in the form of functions called the prior distribution functions, and these functions express the prior knowledge of the parameters and when r is taken from n from Units subject to the Topp Leone distribution with the parameter λ and based on type II censor data and assuming that the initial distribution of the parameter λ is a gamma distribution :

$$h(\lambda;\delta,\gamma)=\frac{\gamma^{\delta}}{\Gamma(\delta)}\lambda^{\delta-1}e^{-\gamma\lambda}\qquad\lambda>0,\delta>0,\gamma>0\ \dots\dots\dots\dots(13)$$

By using the Bayesian inversion formula and the maximum possibility function for observations in Equation 10, the posterior distribution function for parameter λ is obtained as follows: [4]

$$R(\lambda,\underline{t}) = \frac{L(\lambda,\underline{t})h(\lambda;\delta,\gamma)}{\int_0^\infty L(\lambda,\underline{t})h(\lambda;\delta,\gamma)d\lambda} \dots\dots\dots (14)$$

$$R(\lambda,\underline{t}) = \frac{(\gamma+N)^{r+\delta}}{\Gamma(r+\delta)}\lambda^{r+\delta-1}e^{-(\gamma+N)\delta} \quad \lambda>0, \delta>0, \gamma>0 \dots\dots\dots (15)$$

1. In order to find a bayesian estimator for the parameter λ, a . of the Bayesian loss function must be found. In our research, the symmetric error squared function will be used, which is defined as follows: [1]

$$L(\hat{\lambda},\lambda) = (\hat{\lambda}-\lambda)^2 \dots\dots\dots (16)$$

Where $\hat{\lambda}$ is nothing but the estimator of the parameter λ where a Bayes estimator for the parameter λ is obtained as follows:

$$\hat{\lambda}_B = E_R(\lambda,\underline{t}) \dots\dots\dots (17)$$

Whereas, $E_R(\lambda,\underline{t})$ is the expectation or mean of the posterior distribution function for parameter λ and is as follows:

$$\hat{\lambda}_B = \frac{r+\delta}{\gamma+N} \dots\dots\dots\dots (18)$$

As for the Bayesian estimator of the survival function for the Topp Leone distribution based on the error-squared loss function, it is as follows [5]

$$\hat{S}_B = E_R[\mathrm{S}(\underline{t},),\underline{t}] \dots\dots\dots.. (19)$$

Relying on equation 15, we get the maximum possibility function of the survival function for the Topp Leone distribution, by performing the following transformation

$$\lambda = \frac{-log\, S(t)}{log(2t-t^2)} \dots\dots\dots (20)$$

Then the maximum possibility function of the survival function for the Top Leone distribution is as follows:

$$Y(S(\underline{t});\underline{t}) = \frac{A}{(log(2t-t^2))^{(r+\delta)}} S(t)^{\gamma+N/log(2t-t^2)}(-lnS(t))^{r+\delta-1} \quad 0<S(t)<1 \dots\dots\dots (21)$$

Whereas:

$$A = \frac{(\gamma + N)^{r+\delta}}{\Gamma(r + \delta)}$$

What is a Bayesian estimator of the survival function of the Topp Leone distribution based on the error-squared loss function is the expectation for the function 21 is as follows:

$\hat{S}_B = E_Y\big(S(t), \underline{t}\big)$ ... ... (22)

Then the Bayesian estimator for the survival function is as follows:

$$\hat{S}_B = \int_0^1 S(t) \frac{A}{(log(2t - t^2))^{(r+\delta)}} S(t)^{\gamma + \frac{N}{log(2t-t^2)}} (-lnS(t))^{r+\delta-1}\, dS(t) \text{ ... ... } (23)$$

$$\hat{S}_B = \left(\frac{\gamma + N}{\gamma + N + log(2t - t^2)}\right)^{r+\delta} \text{ ... ... } (24)$$

3.3 E-Bayesian estimation method

The first to introduce this method is the researcher Han in 2009 [6], where he assumed that the hyper parameters can be made from being fixed to being random variables that have a probabilistic function. To be estimated must be decreasing, so the meta-parameters that fulfill this condition must be chosen and by referring to equation 13 and when deriving it for parameter λ, we get the following equation [2]

$$\frac{dh(\lambda; \delta, \gamma)}{d\lambda} = \frac{\gamma^{\delta}}{\Gamma(\delta)} \lambda^{\delta-2}\, e^{-\gamma\lambda} \left[(\delta - 1) - \gamma\lambda\right] \text{ ... ... } (25)$$

In order for the function $h(\lambda; \delta, \gamma)$ to be decreasing to λ, and from noting the derivative in equation 24, it must be $0 < \delta < 1$ and $\gamma > 0$. Assuming that the meta-parameters δ, γ are independent, the joint distribution of the two parameters is as follows:

$$\alpha(\delta, \gamma) = \alpha(\delta)\alpha(\gamma)$$

Assuming that the initial probability distribution functions for the meta-parameters δ, γ are given as follows [9]

$$\alpha_1(\delta, \gamma) = \frac{1}{m\beta(a, b)} \delta^{a-1}(1 - \delta)^{b-1} \quad 0 < \delta < 1, 0 < \gamma < m \text{ ... ... } (26)$$

$$\alpha_2(\delta, \gamma) = \frac{2}{m^2\beta(a, b)} (m - \gamma)\delta^{a-1}(1 - \delta)^{b-1} \quad 0 < \delta < 1, 0 < \gamma < m \text{ ... ... } (27)$$

$$\alpha_3(\delta,\gamma) = \frac{2\gamma}{m^2\beta(a,b)}\delta^{a-1}(1-\delta)^{b-1} \qquad 0<\delta<1, 0<\gamma<m \dots\dots\dots.(28)$$

Whereas :$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is Beta function

And that the constant m is only the upper bound for the parameter γ and is very close to the parameter δ in order to ensure the immunity of the Bayesian estimator [2].

The Bayesian expectation estimator for the parameter λ can be calculated as:

$$\hat{\lambda}_{EBi=} \iint\limits_{\forall} \hat{\lambda}_B\,(\delta,\gamma)\alpha_i(\delta,\gamma)d\gamma d\delta \quad i = 1,2,3 \quad \dots\dots\dots(29)$$

$$\hat{\lambda}_{EB1=} \int_0^1 \int_0^m \hat{\lambda}_B\,\alpha_1(\delta,\gamma)d\gamma d\delta$$

$$\hat{\lambda}_{EB1=} = \int_0^1 \int_0^m \frac{r+\delta}{\gamma+N}\frac{1}{m\beta(a,b)}\delta^{a-1}(1-\delta)^{b-1}d\gamma d\delta$$

$$\hat{\lambda}_{EB1=} = \frac{1}{m}\left(r+\frac{a}{a+b}\right) ln[\frac{m+N}{N}] \dots\dots\dots\dots.(30)$$

According to the two formulas (27) (28) for the initial distribution of the meta-parameters (δ, γ), and respectively, we find the estimator of the Bayesian expectation for the parameter λ and as follows

$$\hat{\lambda}_{EB2} = \int_0^1 \int_0^m \frac{r+\delta}{\gamma+N}\frac{2}{m^2\beta(a,b)}(m-\gamma)\delta^{a-1}(1-\delta)^{b-1}d\gamma d\delta$$

$$\hat{\lambda}_{EB2} = \frac{2}{m}\left(r+\frac{a}{a+b}\right)[\frac{N+m}{m}\ln\left(\frac{N+m}{N}\right)-1] \dots\dots\dots\dots.(31)$$

$$\hat{\lambda}_{EB3} = \int_0^1 \int_0^m \frac{r+\delta}{\gamma+N}\frac{2\gamma}{m^2\beta(a,b)}\delta^{u-1}(1-\delta)^{v-1}d\gamma d\delta$$

$$\hat{\lambda}_{EB3} = \frac{2}{m}\left(r+\frac{a}{a+b}\right)[1-\frac{N}{m}\ln\left(\frac{N+m}{N}\right)] \dots\dots\dots\dots.(32)$$

4-3 E-Bayesian estimation for survival function for Topp Leone Distribution

The survival function of the Topp Leone distribution will be estimated based on the initial distributions of the meta-parameters (δ, γ), which were previously discussed in equations (26) (27) (28), as well as on the squared error loss function. Then, the Bayesian expectation estimator for a retention function of the Topp leone distribution can be found as follows

$$\hat{S}_{EBi} = \int_0^1 \int_0^m \hat{S}_B(t)\alpha_i(\delta,\gamma)d\gamma d\delta \qquad i = 1,2,3 \quad \dots\dots\dots(33)$$

$$\hat{S}_{EB1} = \int_0^1 \int_0^m (\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^{r+\delta} \frac{1}{m\beta(a,b)} \delta^{a-1}(1-\delta)^{b-1} \quad d\gamma d\delta$$

$$= \frac{1}{m\beta(a,b)} \int_0^m (\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^{r} [\int_0^1 e^{\delta log(\frac{\gamma+N}{\gamma+N+log(2t-t^2)})} \delta^{a-1}(1-\delta)^{b-1}\, d\delta]\, d\gamma$$

$$\int_0^1 e^{\delta log(\frac{\gamma+N}{\gamma+N+log(2t-t^2)})} \delta^{a-1}(1-\delta)^{b-1}\, d\delta = F(a, a+b;\ log\left(\frac{\gamma+N}{\gamma+N+log(2t-t^2)}\right) \dots\dots\dots.(34)$$

Whereas

$$\int_0^u x^{v-1}(u-x)^{m-1}e^{bx}dx = \beta(m,v)u^{m+v-1}1F1(v, m+v, \lambda u) \dots\dots..(35)$$

$$\hat{R}_{EBs1} = \frac{1}{m}\int_0^m (\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^{r} F(a, a+b; log\left(\frac{\gamma+N}{\gamma+N+log(2t-t^2)}\right)\ d\gamma \dots\dots\dots(36)$$

Where 1F1(v,m+v,λu) is the Hyper geometric function Generalized [8], the formula 3.383(1)

And based on the primary distribution functions for the two parameters (δ, γ) mentioned in equations (27) (28), respectively, and in the same previous method, we will find the biological expectancy estimator for the survival function of the Topp Leone distribution.

$$\hat{S}_{EB2} = \int_0^1 \int_0^m (\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^{r+\delta} \frac{2}{m^2\beta(a,b)} (m-\gamma)\delta^{a-1}(1-\delta)^{b-1}\, d\gamma d\delta$$

$$= \frac{2}{m^2\beta(a,b)} \int_0^m (m - \gamma)(\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^{r} [\int_0^1 e^{\delta log\left(\frac{\gamma+N}{\gamma+N+log(2t-t^2)}\right)} \delta^{a-1}(1-\delta)^{b-1}\, d\delta] d\gamma$$

$$\hat{S}_{EB2} = \frac{2}{m^2}\int_0^m (m-\gamma)(\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^r F\left(a,a+b;log\left(\frac{\gamma+N}{\gamma+N+log(2t-t^2)}\right)\right) d\gamma \ldots\ldots\ldots(37)$$

$$\hat{S}_{EB3} = \int_0^1\int_0^m (\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^{r+\delta}\frac{2\gamma}{m^2\beta(a,b)}\delta^{a-1}(1-\delta)^{b-1} \qquad d\gamma d\delta$$

$$= \frac{2\gamma}{m^2\beta(a,b)}\int_0^m (\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^r[\int_0^1 e^{\delta log\left(\frac{\gamma+N}{\gamma+N+log(2t-t^2)}\right)}\delta^{a-1}(1-\delta)^{b-1}\,d\delta]d\gamma$$

$$\hat{S}_{EB3} = \frac{2\gamma}{m^2}\int_0^m (\frac{\gamma+N}{\gamma+N+log(2t-t^2)})^r F\left(a,a+b;log\left(\frac{\gamma+N}{\gamma+N+log(2t-t^2)}\right)\right) d\gamma \ldots\ldots\ldots(38) \;.$$

Using numerical methods, it is possible to find the results of the integrations of equations (36) (37) (38) to obtain the Bayesian expectation estimator for the survival function of the Topp Leone distribution and based on the primary distribution functions of the two parameters (δ, γ) contained in equations (26) (27) (28)) *respectively*.

4-Simulation

To estimate the shape parameter in the Topp Leone distribution, using the R program, we follow the following stages
The first stage: here the default sampling values, the monitored sampling sizes n, and the failure sampling sizes r are set as follows
We generate the values of the meta-parameters a, b and as follows:

$$\gamma \sim beta(a,b) \qquad ,\delta \sim uniform(0,m)$$

Assume that: $m = 0.1 \quad , a = 2, b = 3$

The second stage: generation of control samples of the second type of the Topp Leone distribution

The third stage: we estimate the parameter λ by the methods mentioned in the theoretical side of the research according to formulas (11)(18)(30)(31)(32).

The fourth stage: estimating the survival function for the Top Lyon distribution, according to the following formulas, the survival function for the Top Lyon distribution, according to the following formulas (12) (24) (36) (37)(38)

The fifth stage: It is the comparison stage in which the comparison will be made between the value of the parameter λ and its estimations according to the methods used in the research and

For an accurate comparison of the priority estimators, the criterion for comparison will be the mean absolute relative error (MAPE).

$$MAPE(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{\lambda - \hat{\lambda}_i}{\lambda}\right|$$

Each experiment was repeated 1000 (N = 1000)

The following tables represent the simulation results

Table (1) Estimated parameter values of λ by different methods when a=2,b=3,m=1

n	r	$\hat{\lambda}_{mle}$	$\hat{\lambda}_{B}$	$\hat{\lambda}_{EB1}$	$\hat{\lambda}_{EB2}$	$\hat{\lambda}_{EB3}$
25	15	0.67213	0.63697	0.63699	0.63698	0.63691
25	20	0.67223	0.63692	0.63698	0.63697	0.63695
25	25	0.67231	0.63259	0.63699	0.63696	0.63692
30	15	0.62946	0.63991	0.63951	0.63950	0.63952
30	20	0.62773	0.63799	0.6375	0.6374	0.6377
30	25	0.62235	0.63121	0.63118	0.63117	0.63118
30	30	0.62231	0.63122	0.63121	0.63120	0.63120
50	30	0.63867	0.6386	0.63844	0.63830	0.638561
50	35	0.63687	0.63673	0.63664	0.63650	0.63666
50	40	0.63644	0.63640	0.63636	0.63631	0.63639
50	45	0.63345	0.63340	0.63332	0.63236	0.63336
50	50	0.63267	0.63258	0.63243	0.63225	0.63247
60	40	0.63977	0.63975	0.63977	0.63975	0.63971
60	45	0.63798	0.63784	0.63775	0.63766	0.63766
60	50	0.63757	0.63751	0.63747	0.63742	0.63741
60	55	0.63456	0.63459	0.63443	0.63445	0.63442

60	60	0.63379	0.63365	0.63366	0.63369	0.63358
70	50	0.64089	0.64021	0.64029	0.64030	0.64031
70	55	0.64215	0.64115	0.64118	0.64119	0.64118
70	60	0.64211	0.64122	0.64117	0.64115	0.64117
70	65	0.64225	0.634121	0.64118	0.64119	0.64116
70	70	0.64231	0.634135	0.64138	0.64140	0.64139

(Table 2) The average absolute relative error (MAPE) of the parameter values λ by the different methods when a=2,b=3,m=1

n	r	$\hat{\lambda}_{mle}$	$\hat{\lambda}_{B}$	$\hat{\lambda}_{EB1}$	$\hat{\lambda}_{EB2}$	$\hat{\lambda}_{EB3}$
25	15	0.0067228	0.0065729	0.0065149	0.0065144	0.0065150
	20	0.0066945	0.0065630	0.0065138	0.0065137	0.0065138
	25	0.0065331	0.0065211	0.0065132	0.0065131	0.0065133
30	15	0.0066892	0.0065423	0.0065141	0.0065140	0.0065142
	20	0.0066731	0.0065410	0.0065140	0.0065136	0.0065141
	25	0.0066220	0.0065203	0.0065138	0.0065135	0.0065139
	30	0.0066217	0.0065198	0.0065134	0.0065132	0.0065135
50	30	0.0066645	0.0065180	0.0065130	0.0065131	0.0065133
	35	0.0066683	0.0065173	0.0065129	0.0065126	0.0065130
	40	0.0066538	0.0065161	0.0065127	0.0065125	0.0065129
	45	0.0066527	0.0065101	0.0065097	0.0065096	0.0065099
	50	0.0066501	0.0065073	0.0065061	0.0065061	0.0065062
60	40	0.0065894	0.0064341	0.0064225	0.0064223	0.0064227
	45	0.0065881	0.0064335	0.0064221	0.0064220	0.0064223
	50	0.0065222	0.0064322	0.0064219	0.0064219	0.0064220
	55	0.0065209	0.0064314	0.0064215	0.0064213	0.0064217
	60	0.0065001	0.0064305	0.0064210	0.0064209	0.0064212
70	50	0.0065049	0.0064111	0.0064097	0.0064095	0.0064098
	55	0.0065027	0.0064105	0.0064083	0.0064080	0.0064082
	60	0.0065011	0.0064090	0.0064056	0.0064055	0.0064057
	65	0.0065019	0.0064072	0.0064043	0.0064041	0.0064043
	70	0.0065001	0.0064053	0.0064012	0.0064011	0.0064013

Table (3) Estimates of Survival Function for Topp Leone Distribution a=2,b=3,m=1

n	r	$\hat{S}_{mle}$	$\hat{S}_{B}$	$\hat{S}_{EB1}$	$\hat{S}_{EB2}$	$\hat{S}_{EB3}$
25	15	0.045006	0.042101	0.042540	0.042542	0.04239
	20	0.046250	0.043182	0.043623	0.043625	0.043620

	25	0.0472717	0.044819	0.044955	0.044957	0.044951
30	15	0.0453510	0.043001	0.044014	0.044021	0.044013
	20	0.047212	0.045107	0.045642	0.045645	0.045640
	25	0.048010	0.045770	0.046081	0.046087	0.046081
	30	0.048257	0.045793	0.046097	0.046099	0.046095
50	30	0.048133	0.047837	0.047925	0.047927	0.047923
	35	0.050400	0.048091	0.048954	0.048958	0.048950
	40	0.050591	0.048207	0.049668	0.048967	0.048966
	45	0.050637	0.048470	0.048980	0.048982	0.048981
	50	0.050704	0.048626	0.049001	0.0489003	0.0489001
60	40	0.051139	0.048000	0.048137	0.048139	0.048135
	45	0.063407	0.049003	0.049060	0.049062	0.049059
	50	0.057248	0.049015	0.049073	0.049075	0.049071
	55	0.059154	0.049041	0.049085	0.049086	0.049083
	60	0.060039	0.049061	0.049096	0.049097	0.049096
70	50	0.061211	0.048911	0.048980	0.048983	0.048980
	55	0.061250	0.048920	0.049025	0.049027	0.049022
	60	0.061420	0.049082	0.049073	0.049075	0.049071
	65	0.061503	0.049095	0.049097	0.049099	0.049099
	70	0.061546	0.049101	0.049111	0.049113	0.049111

4-Conclusion and recommendations

1. We recommend that analysts in the fields of reliability survival analysis and engineering utilize the E-Bayesian (Expected-Bayesian) method when estimating the parameters and functions of the Topp-Leone distribution, given its high efficiency and error-reduction capabilities.

2. We recommend extending the application of the E-Bayesian method to estimate the reliability of complex systems (such as parallel and series systems) in empirical and field-based studies.

3. It is suggested that future research apply these estimation methods to the Topp-Leone distribution using alternative censoring schemes, such as Progressive Type-II Censoring or Hybrid Censoring.

4. We recommend studying the E-Bayesian estimation of the Topp-Leone distribution under asymmetric loss functions, such as the General Entropy or the Linex loss function, and comparing the results with the symmetric squared error loss function utilized in this study.

5. We notice a decrease in AMPE with an increase in the sample size n and an increase in the failure sample size r, as the Bayesian . estimators have high efficiency.

6. In general, we find that the Bayesian estimator and the maximum possibility estimator for the parameter λ and the survival function are greater than the Bayesian expectation estimator

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